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LETTER TO THE EDITOR

A new approximate stationary probability distribution for processes driven by Ornstein–Uhlenbeck noise

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Abstract. A novel approximate probability distribution for a class of non-linear processes driven by multiplicative linear Ornstein–Uhlenbeck noise is presented. A comparison is made with other approximate distributions reported in the literature.

In a recent paper [1] one of the authors derived an approximate evolution equation for a one-dimensional probability distribution $p(x, t)$ of the stochastic process x_t modelled by the equation

$$\dot{x}_t = f(x_t) + \lambda g(x_t)\xi(t) \qquad x \in (x_1, x_2) \tag{1}$$

where $\xi(t)$ is the Ornstein–Uhlenbeck noise [2]

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\xi(s) \rangle = D/\tau \exp(-1/\tau|t-s|).$$

The positive parameters D and τ are the intensity and correlation time of $\xi(t)$, respectively. The parameter λ is a 'coupling constant'.

The evolution equation for $p(x, t)$ is not of Fokker–Planck type but has the form [1]

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x} f(x) p(x,t) - \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} \int_{x_0}^{x} \mathrm{d}x'\, \mathbb{W}_t(x, x') p(x', t) \tag{2}$$

where

$$\mathbb{W}_t(x, x') = \frac{\Omega(x', t)}{\Lambda(x', t)} \exp\left(\lambda^{-2} \int_{x'}^{x} \frac{\mathrm{d}u}{\Lambda(u, t)} \right)$$

$$\Omega(x, t) = \frac{D}{\tau} \int_0^t \mathrm{d}s\, \mathrm{e}^{-s/\tau} F(x, s)$$

$$\Lambda(x, t) = \frac{D}{\tau} \int_0^t \mathrm{d}s\, \mathrm{e}^{-s/\tau} \int_0^s \mathrm{d}u\, \mathrm{e}^{u/\tau} [F'(x, s)F(x, u) - F(x, s)F'(x, u)]$$

($F'(x, t)$ denotes the derivative of $F(x, t)$ with respect to x) and the function $F(x, t)$ is a solution of the equation

$$\frac{\partial F(x,t)}{\partial t} = f'(x) F(x,t) - f(x) \frac{\partial F(x,t)}{\partial x} \qquad F(x, 0) = g(x).$$

The lower limit of integration x_0 in (2) is either x_1 or x_2 (cf equation (1)) and is determined by the fact that in some limiting cases equation (2) should be reduced to an equation of Fokker-Planck type [1]. Equation (2) was derived under the assumption that λ is 'small'.

If the limits

$$\lim_{t\to\infty} \Omega(x, t) = \Omega(x) \qquad \lim_{t\to\infty} \Lambda(x, t) = \Lambda(x) \tag{3}$$

exist then the steady-state solution $p_s(x)$ of equation (2) has the form

$$p_s(x) = \frac{Ng(x)}{g(x)\Omega(x)+f(x)\Lambda(x)} \exp\left(\lambda^{-2} \int \mathrm{d}x \frac{f(x)}{g(x)\Omega(x)+f(x)\Lambda(x)}\right) \tag{4}$$

where N is the normalisation constant.

Let us consider a special case of equation (1) with

$$f(x) = ax - bx^{\gamma+1} \qquad g(x) = cx \qquad x \in [0, \infty) \tag{5}$$

where γ, a and b are positive constants, and c is an arbitrary constant. Applications of equation (1) with (5) are presented in [3-11] and references therein.

For the model (5) functions $\Omega(x)$ and $\Lambda(x)$ in (3) have the form [1]

$$\Omega(x) = cDx\left(1 - \frac{\gamma b\tau}{1+\gamma a\tau} x^{\gamma}\right) \qquad \Lambda(x) = -c^2 D \frac{\gamma b \tau}{a(1+\gamma a\tau)} x^{\gamma+1}.$$

The stationary probability distribution (4) becomes

$$p_s(x) = Nx^{-1+a/Dc^2}(Ax^{2\gamma} - Bx^{\gamma}+1)^{-1-a/2\gamma Dc^2}$$
$$\times \exp\left\{\frac{-1}{\gamma Dc^2}\left(\frac{a}{\gamma\tau}\right)^{1/2} \tan^{-1}\left[\left(\frac{\gamma\tau}{a}\right)^{1/2}(bx^{\gamma}-a)\right]\right\} \tag{6}$$

where

$$A = \frac{\gamma\tau b^2}{a(1+\gamma a\tau)} \qquad B = \frac{2\gamma b\tau}{1+\gamma a\tau}.$$

In equation (6) we have put $\lambda = 1$ since λ occurs only in the combination $\lambda^2 D$ (then small λ is equivalent to small D). In figure 1 we present representative shapes of $p_s(x)$.

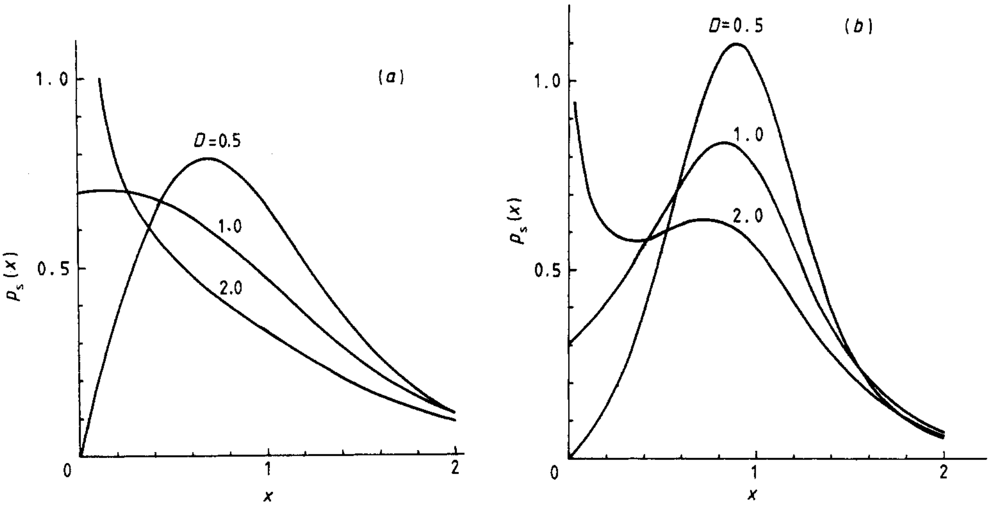
The function in the denominator of (6) is positive for all finite values of the correlation time τ. So, $p_s(x)$ is certainly definite on the whole phase space $[0, \infty)$ of the system (5). The form of $p_s(x)$ is rather non-typical. The normalisation of $p_s(x)$ is guaranteed by this denominator. Usually, the normalisation of the stationary distribution is guaranteed by a factor which is an exponential function of a polynomial in x [4, 7, 9, 10] and this exponential function tends to zero as x tends to infinity. The exponential function in (6) does not tend to zero as x tends to infinity.

Now, we can compare our solution (6) with other solutions reported. The small-noise theories give (cf equations (2.36) and (2.37) in [4] and (2.17) and (2.15) in [6])

$$P_{1s}(x) = N_1 x^{-1+a/Dc^2}\left(1 - \frac{\gamma b\tau}{1+\gamma a\tau} x^{\gamma}\right)^{-1+1/\tau D\gamma^2 c^2}. \tag{7}$$

The function $P_{1s}(x)$ is definite only on the interval $[0, x_c]$, where x_c is determined by the equation

$$\gamma b\tau x_c^{\gamma} = 1 + \gamma a\tau.$$



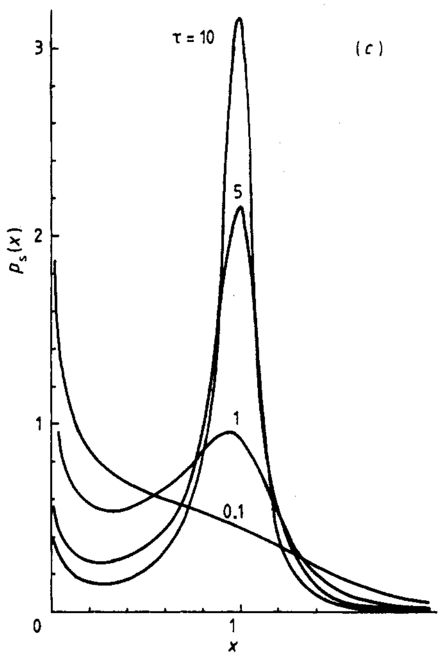


Figure 1. Stationary distributions given by equation (6) for: (a) $a=b=c=2$, $\gamma=1$, $\tau=0.2$ and various D; (b) $a=b=c=2$, $\gamma=1$, $\tau=1$ and various D; (c) $a=b=c=1$, $\gamma=2$, $D=1.5$ and various τ. Qualitatively, the same shapes as in (a) are obtained in the white-noise case ($\tau=0$). In (b) and (c) new features occur, which are absent in the white-noise case.

The conventional small-correlation-time theory [4, 5] yields (cf equation (2.23) in [4])

$$P_{2s}(x)=N_2x^{-1+a/Dc^2}\left\{1-\tau\left[\frac{\gamma}{2}+\frac{a^2}{2Dc^2}-b\left(\gamma+\frac{a}{Dc^2}\right)x^\gamma+\frac{b^2}{2Dc^2}x^{2\gamma}\right]\right\}\exp\left(-\frac{bx^\gamma}{\gamma Dc^2}\right) \tag{8}$$

and the Fox small-correlation-time theory [8, 10] leads to

$$P_{3s}(x)=N_3x^{-1+a/Dc^2}(1+\gamma b\tau x^\gamma)\exp\left[\frac{1}{Dc^2}\left(\frac{b}{\gamma}(\gamma a\tau-1)x^\gamma-\frac{1}{2}\tau b^2x^{2\gamma}\right)\right]. \tag{9}$$

Here, N_i ($i = 1, 2, 3$) are the normalisation constants. We do not present the stationary distribution which can be obtained from the Altares and Nicolis theory [9] because their theory is local in space around one steady state. Their theory gives constructive results concerning the most probable value x_m of the process x_t. From equations (6)–(9) we can obtain the most probable values x_m of the process which are determined by the extrema of the stationary distributions. One can check that each theory gives different values of x_m.

The distribution (7) cannot be acceptable for two reasons. It is not definite on the whole phase space $[0, \infty)$ and for some values of the noise parameters it leads to predictions that do not agree with either experiments or other theories (see, e.g., figure 6 in [6]). The distribution (8) cannot be acceptable from a fundamental point of view. It is not positive definite on the whole phase space $[0, \infty)$. It becomes negative for sufficiently large x. Only the distribution (9) is well behaved. It is defined and positive definite on the whole phase space.

Let us note the following fact. If we apply the exponentiation procedure (i.e. the transformation of the first terms of a Taylor expansion into an exponential, see section IIIB of [4]) to the distribution (8) then (cf equation (3.12) in [4])

$$\tilde{P}_{2s}(x) = \tilde{N}_2 x^{-1+a/Dc^2} \exp\left[\frac{1}{Dc^2}\left(\frac{b}{\gamma}(\gamma a\tau - 1)x^\gamma - \frac{1}{2}\tau b^2 x^{2\gamma}\right)\right] \exp(\gamma b\tau x^\gamma). \tag{10}$$

The first-order τ expansion of the last exponential function in (10) yields $P_{3s}(x)$, equation (9). Thus, from (8) we can obtain (9) by use of a somewhat artificial procedure.

In figure 2 we compare our result (6) with the distribution (9). It is seen that both our approximation and that of Fox predict qualitatively the same main features of model (5). There is a good quantitative agreement between (6) and (9) for small τ. For some values of the noise parameters, a quantitative agreement of the simulation results [4] with (9) is better than with (6), but this should be expected since the accuracy of each theory considered is limited by its assumptions (small intensity or/and small correlation time of the noise).

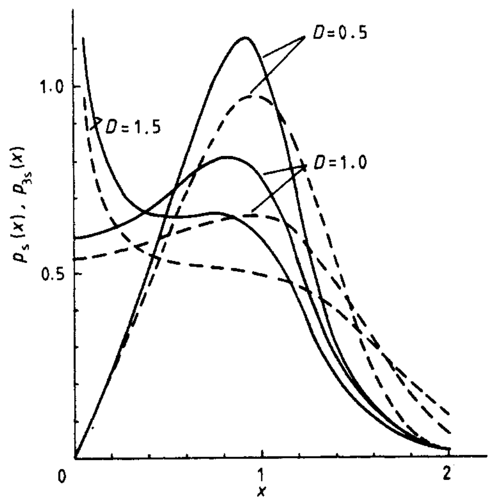


Figure 2. The comparison of distributions $p_s(x)$ given by equation (6) (full curves) and $p_{3s}(x)$ given by (9) (broken curves) for $a = b = c = 1$, $\gamma = 2$, $\gamma = \frac{1}{3}$ and various D.

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